# The Stanley decomposition of the harmonic oscillator

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Communicated by Prof. J.J. Duistermaat at the meeting of March 28, 1988

ABSTRACT

This paper gives a new decomposition for the ring of polynomial functions on the variety of $(n+1)\times(n+1)$ complex matrices of rank less than or equal to one. This involves decomposing the monoid

$$\mathscr{M}_n=\{(j,k)\in\mathbb{N}^{n+1}\times\mathbb{N}^{n+1}\,|\,|j|=|k|\}$$

into a finite disjoint union of translates of $\mathbb{N}$ cones based on certain $2n$ simplices in $\mathbb{R}^{2n+2}$. As a consequence we have a method for writing the normal form of a perturbed $n+1$ dimensional harmonic oscillator in a unique way.

## 1. INTRODUCTION

Let $\mathscr{A}_n$ be the ring of polynomial functions on the $2n$ dimensional complex affine variety $M_{n+1,l}$ of $(n+1)\times(n+1)$ complex matrices of rank less than or equal to one. The main point of this article is to give a new decomposition of $\mathscr{A}_n$, which we call the Stanley decomposition.

In order to describe this decomposition we need a more explicit representation of the ring $\mathscr{A}_n$. In § 5 we show that $\mathscr{A}_n$ is isomorphic to the ring

$$\mathscr{B}_n=\mathbb{C}[\mathbb{C}^{n+1}\times(\mathbb{C}^{n+1})^*]^{\mathbb{C}^*}$$

of polynomials on $\mathbb{C}^{n+1}\times(\mathbb{C}^{n+1})^*$ which are invariant under the $\mathbb{C}^*$ action

$$(t,(x_0,\dots,x_n,y_0,\dots,y_n))\mapsto(tx_0,\dots,tx_n,t^{-1}y_0,\dots,t^{-1}y_n).$$

Partly supported by the U.S. Army Research Office through the Mathematical Sciences Institute of Cornell University.

Clearly the monomials $\pi_{ij}=x_i y_j$, $0 \leqq i, j \leqq n$ are invariant under the $\mathbb{C}^*$ action. In fact $\{\pi_{ij}\}$ generate $\mathscr{B}_n$. This is shown as follows. Using multi-index notation, the monomial $M=x^i y^j$ is invariant under $\mathbb{C}^*$ iff for all $t \in \mathbb{C}^*$ $M=t^{|i|-|j|}M$, that is, iff

$$|i|=i_0+\cdots+i_n=|j|=j_0+\cdots+j_n.$$

Writing the factors of an invariant monomial as two lists

$$\text{(1a)} \qquad \overbrace{x_0,\ldots,x_0}^{i_0}, \overbrace{x_1,\ldots,x_1}^{i_1},\ldots,\overbrace{x_n,\ldots,x_n}^{i_n}$$

$$\text{(1b)} \qquad \underbrace{y_0,\ldots,y_0}_{j_0}, \underbrace{y_1,\ldots,y_1}_{j_1},\ldots,\underbrace{y_n,\ldots,y_n}_{j_n}.$$

which have the same number of entries since $|i|=|j|$, and pairing off the corresponding entries, shows that an invariant $M$ is a product of suitable $\pi_{ij}$. Clearly we have the relations

$$\text{(2)} \qquad \pi_{ij}\pi_{kl}=\pi_{il}\pi_{kj}$$

for all $0 \leqq i, j, k, l \leqq n$. In § 5 it is shown that these are the only relations among the generators of $\mathscr{B}_n$.

Now we are in a position to describe the Stanley decomposition of $\mathscr{B}_n$. Consider the $\mathbb{N}$ monoid

$$\mathscr{M}_n=\{(i,j)\in \mathbb{N}^{n+1}\times\mathbb{N}^{n+1} \mid |i|=|j|\}$$

which is the exponent monoid of $\mathscr{B}_n$. By a theorem of Stanley [13, p. 191], $\mathscr{M}_n$ can be written as a finite disjoint union of translates of $N$ commutative free $\mathbb{N}$ monoids $K_{\sigma_l}$ on $2n+1$ generators (see Theorem 3, § 3). In symbols,

$$\text{(3)} \qquad \mathscr{M}_n=\coprod_{l=1}^{N}(\eta_l+K_{\sigma_l}).$$

From the decomposition of $\mathscr{M}_n$ follows the Stanley decomposition of $\mathscr{B}_n$; namely,

$$\text{(4)} \qquad \mathscr{B}_n=\sum_{l=1}^{N}\oplus x^{\alpha_l}y^{\beta_l}\mathbb{C}[\{\pi_{rs} \mid (r,s)\in\sigma_l\}]$$

where $\eta_l=(\alpha_l,\beta_l)\in\mathbb{N}^{n+1}\times\mathbb{N}^{n+1}$. And conversely, from the Stanley decomposition of $\mathscr{B}_n$ one obtains the decomposition of $\mathscr{M}_n$. Unfortunately Stanley's proof of the decomposition (3) is nonconstructive. What we do in this paper is to give an explicit constructive proof of (4).

The precise algebraic-geometric meaning of (4) is unknown to the authors. Algebraically seen the Stanley decomposition of $\mathscr{B}_n$ is, on the one hand, weaker than the statement that $\mathscr{B}_n$ is Cohen-Macaulay (namely, there are elements $z_1,\ldots,z_r\in\mathscr{B}_n$ called polynomial generators and elements $\eta_1,\ldots,\eta_s$ called separators such that

$$\mathscr{B}_n=\sum_{l=1}^{s}\oplus\eta_l\mathbb{C}[z_1,\ldots,z_r])$$

because the choice of polynomial generators depends on the summand. On the other hand, the Stanley decomposition is stronger than Cohen-Macaulay because the separators are monomials.

We end this introduction by writing down the Stanley decomposition of $\mathscr{B}_2$. Given a monomial $M = x^i y^j$ with

$$|i| = i_0 + i_1 + i_2 = j_0 + j_1 + j_2 = |j|,$$

using the relations (2) it can be straightened into a unique monomial

$$\tilde{M} = (x_{i_1}, y_{j_1})(x_{i_2}, y_{j_2}) \cdots (x_{i_l}, y_{j_l})$$

(written also as $(i_1, j_1)(i_2, j_2) \cdots (i_l, j_l)$) where

(5) $\qquad i_1 \leqq i_2 \leqq \cdots \leqq i_l$ and $j_1 \leqq j_2 \leqq \cdots \leqq j_l$

(see § 2). Plotting the nodes $(i, j)$ of $\tilde{M}$ as in Figure 1, we see that (5) means that no node $(i_k, j_k)$ in $\tilde{M}$ lies strictly above and strictly to the right of any

| $(i,j)$ | 0 | 1 | 2 | $j$ |
|---|---|---|---|---|
| 0 | • | | | |
| 1 | | | | |
| 2 | | • | • | |
| $i$ | | | | |

Figure 1. Nodes of the monomial $(x_0 y_0)^2 (x_2 y_1)(x_2 y_2)^2$

other node of $\tilde{M}$. Thus all nodes of $\tilde{M}$ lie on a **maximal monotone path,** i.e. a path beginning at (0, 0) and ending at (2, 2) which is made up of moves to the right or moves down. The maximal monotone paths of Figure 1 are given in Figure 2 below.

$$\begin{array}{l}
(0,0) \\
\quad | \\
(1,0) \\
\quad | \\
\underline{(2,0)}\text{–}(2,1)\text{–}(2,2)
\end{array}
\qquad
\begin{array}{l}
(0,0) \\
\quad | \\
\underline{(1,0)}\text{–}(1,1) \\
\qquad\qquad | \\
\qquad\quad \underline{(2,1)}\text{–}(2,2)
\end{array}$$

$$\sigma_1 \qquad\qquad\qquad \sigma_2$$

$$\begin{array}{l}
(0,0) \\
\quad | \\
\underline{(1,0)}\text{–}(1,1)\text{–}(1,2) \\
\qquad\qquad\qquad\quad | \\
\qquad\qquad\qquad (2,2)
\end{array}
\qquad
\begin{array}{l}
(0,0)\text{–}(0,1) \\
\qquad\qquad | \\
\qquad\quad (1,1) \\
\qquad\qquad | \\
\qquad\quad \underline{(2,1)}\text{–}(2,2)
\end{array}$$

$$\sigma_3 \qquad\qquad\qquad \sigma_4$$

$$
\begin{array}{ccc}
(0,0)\text{–}(0,1) & & \\
| & & \\
\underline{(1,1)}\text{–}(1,2) & & \\
 & | & \\
 & (2,2) &
\end{array}
\qquad\qquad
\begin{array}{c}
(0,0)\text{–}(0,1)\text{–}(0,2) \\
\hfill | \\
\hfill (1,2) \\
\hfill | \\
\hfill (2,2)
\end{array}
$$

$$\sigma_5 \qquad\qquad\qquad \sigma_6$$

Figure 2. Maximal monotone paths. Nodes of $\sigma_l$ corresponding to corners are underlined.

A **corner** of a maximal monotone path is a node which is arrived at by a move downward and left by a move to the right. Corners in Figure 2 are the nodes which are underlined. The Stanley decomposition of $\mathscr{B}_2$ is

$$
\begin{aligned}
\mathscr{B}_2 = &(x_2 y_0)\mathbb{C}[x_0 y_0, x_1 y_0, x_2 y_0, x_2 y_1, x_2 y_2] \\
&\oplus (x_1 y_0)(x_2 y_1)\mathbb{C}[x_0 y_0, x_1 y_0, x_1 y_1, x_2 y_1, x_2 y_2] \\
&\oplus (x_1 y_0)\mathbb{C}[x_0 y_0, x_1 y_0, x_1 y_1, x_1 y_2, x_2 y_2] \\
&\oplus (x_2 y_1)\mathbb{C}[x_0 y_0, x_0 y_1, x_1 y_1, x_2 y_1, x_2 y_2] \\
&\oplus (x_1 y_1)\mathbb{C}[x_0 y_0, x_0 y_1, x_1 y_1, x_1 y_2, x_2 y_2] \\
&\oplus \mathbb{C}[x_0 y_0, x_0 y_1, x_0 y_2, x_1 y_2, x_2 y_2].
\end{aligned}
$$

The monomials $x_i y_j$ which are polynomial generators in a given summand correspond to the nodes $(i, j)$ in a maximal monotone path. The separators are the products of monomials corresponding to the corners of the monotone path.

## 2. STANDARD MONOMIALS AND THE STRAIGHTENING PROCESS

In this section we give a proof of the Stanley decomposition for the algebra $\mathscr{B}_n$ using the technique of straightening. A similar proof is given in [5] for the Stanley decomposition of the algebra of polynomials on the Grassmannian of 2-planes in $n$ space.

Associate to the monomial $\pi_{jk}$ the **bracket** $(j, k)$. We say that $(j, k) < (j', k')$ if $k < k'$ or, if $k = k'$, $j < j'$. This gives a complete ordering on the brackets. We associate to the monomial

$$M = \pi_{i_1 j_1}\pi_{i_2 j_2}\cdots\pi_{i_r j_r} \quad 0 \leqq i_l, j_l \leqq n \text{ for } l = 1, \ldots, r$$

the **bracket monomial**

$$M = (i_1, j_1)(i_2, j_2)\cdots(i_r, j_r),$$

which we also write as a 2 column tableau

$$M = \begin{pmatrix} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_r & j_r \end{pmatrix}. \tag{6}$$

Here we use the following convention on the indices:

$$(i_1, j_1) \leqq (i_2, j_2) \leqq \cdots \leqq (i_r, j_r).$$

The number $r$ is called the **length** of $M$, i.e. $l(M) = r$. The pair $(i_r, j_r)$ is called the **height** of $M$, i.e. $ht(M) = (i_r, j_r)$.

Thus the $\mathbb{Z}$ algebra of $\mathbb{C}^*$ invariant polynomials is the bracket algebra **B** generated by the brackets $(j, k)$ where $1 \leqq j, k \leqq n$. The relations (2) become the basic syzygy

$$(7) \qquad (j,k)(l,m) = (j,m)(l,k), \quad 1 \leqq j,k,l,m \leqq n$$

among the brackets. In terms of $2 \times 2$ tableaux, (7) reads

$$(8) \qquad \begin{pmatrix} j & k \\ l & m \end{pmatrix} = \begin{pmatrix} l & k \\ j & m \end{pmatrix}$$

DEFINITION. A tableau (6) is called a **standard tableau** if $i_1 \leqq i_2 \leqq \cdots \leqq i_r$.

If we represent $(j, k)$ as nodes in $\mathbb{Z}^2 \cap ([1, n] \times [1, n])$, then the nodes $\{(j_l, k_l) \mid 1 \leqq l \leqq r\}$ represent a standard tableau if no $(j_l, k_l)$ lies strictly above and strictly to the right of a $(j_{l'}, k_{l'})$. An arbitrary tableau can be **straightened** into a standard tableau by applying the basic syzygy (8). This is the content of the following

LEMMA 1. The $\mathbb{Z}$ span of the standard tableau is the bracket algebra **B**.

PROOF. We prove this by induction on the length of $M$. The statement is trivial for $l(M) = 1$. Suppose we can bring tableaux of length $\leqq k$ into standard form. Take $M$ with $l(M) = k + 1$. Define $M' = M/ht(M)$. Since $l(M') < l(M)$ we can bring $M'$ into standard form $\tilde{M}'$. Consider the $2 \times 2$ tableau consisting of $ht(\tilde{M}')$ and $ht(M)$. Let us write this as

$$(9) \qquad \begin{pmatrix} j' & k' \\ j & k \end{pmatrix}$$

There are two possibilities:

1 Either $j' \leqq j$, in which case $M$ is standard.

2 Or $j' > j$ in which case using (8) we write (9) as

$$\begin{pmatrix} j & k' \\ j' & k \end{pmatrix}$$

If we do this to $M$ to get $\tilde{M}$, $ht(\tilde{M}) > ht(M)$.

Since we can increase the height of $M$ only a finite number of times, at a certain moment after repeating the whole procedure a number of times we must end up with case 1. Then we are done. ■

LEMMA 2. The standard tableaux are linearly independent.

PROOF. Among all the nontrivial linear dependence relations among the standard tableaux $\{M_k\}_{k=1}^{N}$ choose one

$$(10) \qquad \sum_{k=1}^{N} c_k M_k = 0$$

with nonzero scalars $c_k$ such that
(a) the number of distinct variables in the $M_k$ is as small as possible;
(b) subject to (a) the maximum length of the tableaux $M_k$ is as small as possible.

Let $(j,k)$ be the maximal bracket (with respect to the height ordering) appearing in (10). By (b) $(j,k)$ is not a common factor of all the $M_k$. Put $(j,k)=0$ in (10). Since the height of $(j,k)$ is maximal, the standard tableaux $M_k$ will remain standard. Thus we obtain a nontrivial relation with fewer variables. This contradicts (a). ■

A **Stanley decomposition** of the space of tableaux $\mathbf{B}$ is a sequence of $m$ free abelian submonoids $\mathbf{B}_1, \ldots, \mathbf{B}_m$ of the multiplicative monoid $\mathbf{B}$, each of rank $k$, and $m$ elements $\eta_1, \ldots, \eta_m \in \mathbf{B}$ such that

$$\mathbf{B} = \sum_{i=1}^{m} \oplus \eta_i \mathbb{Z}[\mathbf{B}_i].$$

Here $\oplus$ is the direct sum of additive abelian groups and $\mathbb{Z}[\mathbf{B}_i]$ is the free additive abelian group generated by $\mathbf{B}_i$.

The goal of this paragraph is to make the Stanley decomposition of $\mathbf{B}$ explicit. Consider the figure

| | 1 | 2 | 3 | ... | $n$ |
|---|---|---|---|---|---|
| 1 | (1, 1) | (1, 2) | (1, 3) | ... | $(1, n)$ |
| 2 | (2, 1) | (2, 2) | (2, 3) | ... | $(2, n)$ |
| . | . | . | . | . | . |
| . | . | . | . | . | . |
| $n$ | $(n, 1)$ | $(n, 2)$ | $(n, 3)$ | ... | $(n, n)$ |

Figure 3.

where each node represents a bracket in $\mathbf{B}$. For each **monotone path** $p_i$ from $(1,1)$ to $(n,n)$, that is, a path which consists of moves downward or to the right, define $\mathbf{B}_i$ to be the free abelian multiplicative monoid generated by the brackets in $p_i$. There are

$$N = \binom{2(n-1)}{n-1}$$

monotone paths. A **corner** of a monotone path is a node which is at the end of a move down and at the start of a move to the right. Thinking of the product of the nodes of a monotone path as a tableau, we order the monotone paths by its inverted lexicographic order: $M<M'$ if $ht(M)<ht(M')$ or, $M/ht(M)<$ $<M'/ht(M')$.

We now prove

THEOREM 1.

$$\mathbf{B}(\{(j,k)\,|\,1\leqq j,k\leqq n)\})=\sum_{l=1}^{N}\eta_l\mathbb{Z}[(r,s)\,|\,(r,s)\text{ is a node of }p_l]$$

where

$$\eta_l=\prod\{(u,v)\,|\,(u,v)\text{ is a corner of }p_l\}.$$

PROOF. By the straightening algorithm, every tableau is a unique linear combination of standard tableaux. Recall that a standard tableau does not have two nodes one of which lies strictly above and to the right of the other. Thus a monotone path $p_l$ corresponds to a standard tableau. Since standard tableaux are linearly independent, it follows that the multiplicative monoid $\mathbf{B}_l$, generated by the nodes of $p_l$, is free. Since $\eta_l\in\mathbb{Z}[\mathbf{B}_l]$, as an additive group $\eta_l\mathbb{Z}[\mathbf{B}_l]$ is generated by standard tableaux.

Because $\mathbf{B}$ is generated as an additive group by the standard tableaux, to prove the theorem it suffices to show that every standard tableau is in a unique $\eta_l\mathbb{Z}[\mathbf{B}_l]$. Let $M$ be a standard tableau. Since $M$ is standard, its factors lie on a monotone path. Let $p$ be the largest monotone path containing the factors of $M$. Suppose $(r,s)$ is a corner of $p$ which is not a factor of $M$. Then $(r-1,s)$ and $(r,s+1)$ are nodes of $p$. Let $q$ be the monotone path created by taking $p$ and replacing $(r,s)$ by $(r-1,s+1)$. Then $q$ still contains the factors of $M$, and $q>p$, since $(r-1,s+1)>(r,s)$ and all other brackets are equal, thus contradicting the maximality of $p$. This proves that to each standard $M$ there is a unique (because it is maximal in the complete ordering) monotone path all of whose corners are factors of $M$. We denote the path by $p_l$ and the product of the corners by $\eta_l$, and we have shown that $M\in\eta_l\mathbb{Z}[\mathbf{B}_l]$. ■

## 3. THE STANLEY DECOMPOSITION OF $\mathscr{M}_n$

Here we give another proof of the Stanley decomposition of the monoid $\mathscr{M}_n$. This proof is more in the spirit of Stanley's original argument [13, p. 191].

We begin with some combinatorial preliminaries.

Let $\Sigma_n$ be an $n$ dimensional simplex with vertices $\{u_0,\dots,u_n\}$. Then the product $\Sigma_n\times\Sigma_n$ has vertices $\{(u_i,u_j)\,|\,i,j\in\{0,\dots,n\}\}$. Consider the vertices of this product as entries in the $(n+1)\times(n+1)$ dimensional array $A_n$. $\Sigma_n\times\Sigma_n$ is a $2n$ dimensional polytope [8]. Therefore a triangulation of $\Sigma_n\times\Sigma_n$ consists of maximal simplices of dimension $2n$, each having $2n+1$ vertices.

A triangulation $\Delta_n$ of $\Sigma_n \times \Sigma_n$ is given in [7, p. 67, def. 8.8]. We will now give another more combinatorial description of this triangulation, due, essentially, to [4]. A **monotone path** in $A_n$ is a sequence of entries of $A_n$ beginning with $(u_0, u_0)$ and ending with $(u_n, u_n)$ such that the entry following $(u_i, u_j)$ is either $(u_i, u_{j+1})$ or $(u_{i+1}, u_j)$. In other words a monotone path is composed of moves either one step to the right or one step down. Each monotone path in $A_n$ consists of $2n+1$ entries. The triangulation $\Delta_n$ of $\Sigma_n \times \Sigma_n$ has maximal $2n$ dimensional simplices $\sigma$ consisting of the convex hull of the vertices corresponding to the entries of the monotone path $p_\sigma$ in $A_n$. An elementary counting argument shows that there are

$$N = \binom{2n}{n}$$

monotone paths in $A_n$. A **corner** of a monotone path is a move down followed by a move to the right. A classical argument, which can be found in [9], shows that the number of monotone paths in $A_n$ which have exactly $i$ corners is

$$h_i = \binom{n}{i}^2.$$

This completes the description of the triangulation of $\Delta_n$.

Here we give another description of $\Delta_n$. Let $\omega$ be the ordering on the vertices $(u_i, u_j)$ of $\Sigma_n \times \Sigma_n$, defined by

$$(u_i, u_j) \leq (u_k, u_l) \text{ if } i \leqq k \text{ and } j \leqq l.$$

Denote the vertex $(u_i, u_j)$ by the node $(i, j)$. The collection of all $d$, $0 \leqq d \leqq 2n$, faces of $\Sigma_n \times \Sigma_n$ is the set of all $d+1$ vertices corresponding to $d+1$ distinct nodes in some monotone path. If $F_d$ is a $d$-face of $\Sigma_n \times \Sigma_n$, let $\delta(F_d)$ be the smallest vertex in $F_d$ in the ordering $\omega$. Thus $\delta(F_d)$ is the upper left most node of the nodes corresponding to $F_d$. We say that the collection of faces $\Phi = (F_0, F_1, \ldots, F_{2n})$ of $\Sigma_n \times \Sigma_n$ is a **full flag** if $\delta(F_i)$ is not a vertex of $F_{i-1}$ for all $i = 1, \ldots, 2n$. If $\Phi$ is a full flag then the $2n+1$ vertices $(\delta(F_0), \delta(F_1), \ldots, \delta(F_{2n}))$ form a $2n$ simplex $\Delta(\Phi)$. Clearly the nodes corresponding to $\delta(F_0), \ldots, \delta(F_{2n})$ form a maximal monotone path. Hence $\Gamma_\omega = \{\Delta(\Phi) \mid \Phi \textit{ is a full flag of } \Sigma_n \times \Sigma_n\}$ form a triangulation of $\Sigma_n \times \Sigma_n$ which is exactly $\Delta_n$.

In order to construct a shelling of the triangulation $\Delta_n$ we must order the maximal $2n$ dimensional simplices $\sigma$ in $\Delta_n$. This we do as follows. Encode the monotone path $p_\sigma$ corresponding to $\sigma$ by a binary string $a_\sigma$: a 0 in the string indicates a move to the right and a 1 a move down. Order the binary strings $a_\sigma$ via the lexicographic order with the string

$$0 \cdots 0 1 \cdots 1$$

first and

$$1 \cdots 1 0 \cdots 0$$

last. Let $\Delta$ be a pure simplicial complex, that is, all the maximal simplices of $\Delta$ have $d+1$ vertices. Then a **shelling** of $\Delta$ is an ordering $\sigma_1, \ldots, \sigma_k$ of all its maximal simplices such that for each $j$, $2 \leqq j \leqq k$,

$$\sigma_j \cap \left( \bigcup_{i<j} \sigma_i \right)$$

is a subcomplex $\Delta'$ of $\Delta$ determined by some number, say $n_j$, of $(d-1)$-faces of $\sigma_j$. When $j=1$, we put $n_1=0$. Having a shelling of $\Delta$ is equivalent to saying that there is a *unique* minimal face $\tau_j$ of $\sigma_j$ which

1. is not in

$$\bigcup_{i<j} \sigma_i$$

and

2. has $n_j$ vertices, each of which is the vertex of $\sigma_j$ omitted from one of the $(d-1)$-faces determining $\Delta'$.

Given a shelling of $\Delta$, define

$$h_i(\Delta) = \#\{j \mid n_j = i\}.$$

The numbers $h_i(\Delta)$ depend only on $\Delta$ and not on the particular shelling (see e.g. [3]). In particular

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(\Delta), \quad 0 \leqq i \leqq d,$$

where $f_j(\Delta)$ is the number of $j$ faces of $\Delta$.

THEOREM 2. The lexicographic order on the $2n$ dimensional simplices of $\Delta_n$ gives a shelling of $\Delta_n$ such that

$$h_i(\Delta) = h_i = \binom{n}{i}^2.$$

In fact for each maximal simplex $\sigma$ of $\Delta$, the minimal face of $\sigma$ not in the union of earlier maximal simplices has vertices corresponding to the corners of the monotone path $p_\sigma$ associated to $\sigma$.

PROOF. Consider the binary string $a_\sigma$ associated to $\sigma$. Each 10 substring corresponds to a corner of $p_\sigma$. Replacing a 10 substring in $a_\sigma$ by a 01 substring results in a binary string $a_{\sigma'}$ where $\sigma' < \sigma$ in the lexicographic order. But $\sigma \cap \sigma'$ is a face of $\sigma$ of codimension 1. Since any $a_{\sigma''}$ for $\sigma'' < \sigma$ is obtained from $a_\sigma$ by a sequence of 10 substring replacements, the vertex set of $\sigma \cap \sigma''$ must lie in $\sigma \cap \sigma'$ where $\sigma'$ is obtained from $\sigma$ by a single 10 substring replacement. The number of such possible replacements in $a_\sigma$ is just the number of corners of $\sigma$. Thus it is the number of codimension 1 faces of $\sigma$ in

$$\sigma \cap \left( \bigcup_{\sigma' < \sigma} \sigma' \right),$$

which in turn is the number of vertices of the unique minimal face $\tau$ of $\sigma$ not in

$$\bigcup_{\sigma'<\sigma} \sigma'.$$

In fact, the unique minimal face $\tau$ has vertices corresponding to the corners of the monotone path $p_\sigma$. ■

In the next paragraphs we construct the Stanley decomposition of $\mathcal{M}_n$. Consider the $\mathbb{N}$ monoid $\mathcal{M}_n$ defined by the set of all $(j,k)\in\mathbb{N}^{n+1}\times\mathbb{N}^{n+1}$ which satisfy

$$(11)\qquad j_0+j_1+\cdots+j_n=k_0+k_1+\cdots+k_n.$$

Let $K_n$ be the cone of nonnegative rational solutions of (11). Slicing $K_n$ with the hyperplane

$$j_0+j_1+\cdots+j_n+k_0+k_1+\cdots+k_n=2$$

gives the polytope $\Sigma_n\times\Sigma_n$. The triangulation $\Delta_n$ of $\Sigma_n\times\Sigma_n$ described in this section gives a decomposition of $K_n$ into $2n+1$ dimensional simplicial cones, that is, cones having $2n+1$ linearly independent extreme rays. Let $K_\sigma$ be the simplicial cone corresponding to the maximal simplex $\sigma\in\Delta_n$.

We now explicitly carry out the decomposition of $\mathcal{M}_n$ into a disjoint union of translates of simplicial cones described by Stanley [13]. More concretely we prove

THEOREM 3.

$$(12)\qquad \mathcal{M}_n=\bigsqcup_{l=1}^{N}(\eta_l+\bar{K}_{\sigma_l}),$$

where $\bar{K}_{\sigma_l}$ is the free commutative $\mathbb{N}$ monoid generated by the vertices of $\sigma_l$; $\sigma_1<\cdots<\sigma_N$,

$$N=\binom{2n}{n}$$

is the shelling of $\Delta_n$; $\eta_l$ is the sum of the vectors associated to the vertices of $\sigma_l$ which correspond to the corners of the path $p_{\sigma_l}$.

On our way toward proving (12) we will need to understand the structure of the integral points of $K_n$ and $K_\sigma$. For this we need the notion of an extreme ray. Corresponding to the entry $(u_r,u_s)$ of $p_\sigma$, $\sigma$ a (maximal) $2n$ simplex in $\Sigma_n\times\Sigma_n$, let $v_{rs}^{(n)}\in\mathbb{R}^{2n+2}$ be the vector with 1 in the $(r+1)^{st}$ and $(n+1+s+1)^{st}$ components and 0 elsewhere. The **extreme rays** of $K_\sigma$ are the vectors $v_{rs}^{(n)}$ where $(u_r,u_s)$ is an entry in the monotone path $p_\sigma$ associated to $\sigma$. We have the

LEMMA 3.
1. Any integral point of $K_n$ lies in some $K_\sigma$.
2. Any integral point of $K_\sigma$ is a unique nonnegative integral combination of the extreme rays of $K_\sigma$.

PROOF. Part (1) has already been proved (see the argument after formula (1)).

To prove part (2) first we note that any such representation is necessarily unique, since the extreme rays of $K_\sigma$ are linearly independent. Next, suppose $z \in K_\sigma \cap \mathbb{Z}^{2n+2}$. We show that any representation

$$z = \sum_{r,s} \lambda_{r,s} v_{r,s}$$

To see this, note that there is a linear order on the $v_{r,s}$ induced by the order of the entries $(u_r, u_s)$ in the path $p_\sigma$. Further, note that each $(u_r, u_\sigma)$ in $p_\sigma$ is either the last entry in row $r$ or the last entry in column $s$. If some $\lambda_{r,s} \notin \mathbb{Z}$, let $(\bar{r}, \bar{s})$ be the first such in this order. Then either entry $\bar{r}+1$ or entry $n+1+\bar{s}+1$ of the vector $z$ is not integral, since for that row or column, $\lambda_{\bar{r},\bar{s}}$ is both the first noninteger weight and the last nonzero weight. Since $z$ is all integer, this is impossible. ■

We remark that a consequence of this lemma is that $\omega$ is a compressed ordering [12]. To see that $\omega$ is compressed it suffices to show that for any rational point $\alpha$ in the interior of a face $F_d$ of $\Sigma_n \times \Sigma_n$, if $c$ is the unique rational number such that

$$\frac{1}{1-c}\,(\alpha - c\delta(F))$$

lies in the boundary of $F_d$ then $l(\alpha)c$ is an integer. Here $l(\alpha)$ is the least common multiple of the denominators of $\alpha$. Write

$$\alpha = \sum_{r=0}^{d} \lambda_r v_r,$$

where $v_r$ are vertices of $F_d$ and $0 < \lambda_r < 1$, $\lambda_r \in \mathbb{Q}$. Then $l(\alpha)\alpha$ is an integral linear combination of extreme rays belonging to some $K_\sigma$. Hence by the lemma, all $l(\alpha)\lambda_r$ are integers. Suppose that $\delta(F_d) = v_s$ for some $s$, $0 \leqq s \leqq d$. Then $\alpha - \lambda_s \delta(F_d)$ belongs to the boundary of $F_d$. Hence $c = \lambda_s$. Therefore $l(\alpha)c$ is an integer.

We now prove theorem 3. Because the geometric nature of $\bar{K}_{\sigma_l}$ has been given by the lemma, we need only show that the $\mathbb{N}$ monoid

$$\mathcal{M}_n = K_n \cap \mathbb{Z}^{2n+2}$$

is a disjoint union given by the statement of the theorem.

We proceed by induction on $N$, the number of maximal simplices in the shelling of $\Delta_n$. Let

$$\Delta^i = \bigcup_{j \leqq i} \sigma_j \text{ and } \bar{K}^i = \bigcup_{j \leqq i} \bar{K}_{\sigma_j}.$$

We have $\Delta_n = \Delta^N$ and $\mathcal{M}_n = \bar{K}_n = \bar{K}^N$. The result is true for $N = 1$ since $\eta_1 = 0$. Suppose that

$$\bar{K}^{i-1} = \bigsqcup_{j<i} (\eta_j + \bar{K}_{\sigma_j}).$$

We know that

$$\bar{K}^i = \bar{K}^{i-1} \cup \bar{K}_{\sigma_i}.$$

Moreover points in

$$\bar{K}_{i-1} \cap \bar{K}_{\sigma_i}$$

are exactly those lying on the $n_i$ codimension 1 subcones of $\bar{K}_{\sigma_i}$ spanned by the $n_i$ codimension 1 faces of $\sigma_i$ which lie in $\Delta^{i-1}$. Therefore points $\bar{K}^i - \bar{K}^{i-1}$ are just those points which do not lie on any of those codimension 1 faces of $\sigma_i$. Therefore $z \in \bar{K}^i - \bar{K}^{i-1}$ is a point whose expression in the monoid $\bar{K}^i$ involves **positive** integer coefficients of each of the corners of $p_{\sigma_i}$. Thus $z \in \eta_i + \bar{K}_{\sigma_i}$. Hence

$$\bar{K}^i = \bar{K}^{i-1} \cup (\eta_i + \bar{K}_{\sigma_i}).$$

No point in $\eta_i + \bar{K}_{\sigma_i}$ can lie in $\bar{K}^{i-1}$ since the corner entries of $p_{\sigma_i}$ span the unique face $\tau_i$ of $\sigma_i$ which does not lie in $\Delta^{i-1}$. Thus the union is disjoint. ■

We note that everything done in the past section extends to the case $\Sigma_r \times \Sigma_s$ corresponding to the monoid $M_{r,s}$ determined by

$$j_0 + j_1 + \cdots + j_r = k_0 + k_1 + \cdots + k_s.$$

The only changes that we have to make are

$$h_i = \binom{r}{i}\binom{s}{i}$$

and

$$N = \binom{r+s}{r}.$$

We end this section by showing that the $h$-vector $(h_0, h_1, \ldots, h_{2n})$ of $\Sigma_n \times \Sigma_n$ where

$$h_i = \binom{n}{i}^2$$

for $0 \leqq i \leqq n$ and $h_i = 0$ for $i > n$ does *not* depend on the triangulation $\Delta_n$ of $\Sigma_n \times \Sigma_n$.

Consider the generating function

$$J(\Sigma_n \times \Sigma_n, t^2) = 1 + \sum_{m \geqq 1} \#\{\alpha \in \Sigma_n \times \Sigma_n \subseteqq \mathbb{R}^{2n+2} \mid m\alpha \in \mathbb{Z}^{2n+2}\} t^{2m}.$$

Since $\Delta_n = \Gamma_\omega$, where $\omega$ is the ordering on the vertices of $\Sigma_n \times \Sigma_n$ defined above and since $\omega$ is compressed, by a theorem of Stanley [12, p. 336] it follows that

$$J(\Sigma_n \times \Sigma_n, t^2) = \frac{\sum_{i=0}^{n} h_i t^{2i}}{(1-t^2)^{2n+1}}$$

where the $h_i$ is the $i$th component of the $h$ vector of the triangulation $\Delta_n$ of $\Sigma_n \times \Sigma_n$.

Now consider the Poincaré series for the algebra $\mathscr{B}_n$ of $\mathbb{C}^*$ invariant polynomials; namely

$$\mathscr{P}\mathscr{B}_n(t) = \sum_{m \geq 0} (\dim_{\mathbb{C}} \mathscr{B}_n^{(m)}) t^{2m},$$

where $\mathscr{B}_n^{(m)}$ is the vectorspace of homogeneous $\mathbb{C}^*$ invariant polynomials of degree $m$. Then

$$(13) \quad \begin{cases} \mathscr{P}\mathscr{B}_n(t) = \sum_{m \geq 0} \#\{(j,k) \in \mathbb{N}^{n+1} \times \mathbb{N}^{n+1} \,\big|\, |j| = |k|, |j| + |k| = 2m\} t^{2m} \\ \qquad = \sum_{m \geq 0} \binom{m+n}{n}^2 t^{2m}, \end{cases}$$

where the last equality follows from the fact that

$$\#\{j \in \mathbb{N}^{n+1} \,\big|\, |j| = m\} = \binom{m+n}{n}.$$

Actually $\mathscr{P}\mathscr{B}_n$ is a rational function in $t^2$ [11]. To find it we substitute the binomial coefficient identity

$$\sum_{j=0}^{n} \binom{N+n+j}{2n} \binom{n}{j} \binom{n}{n-j} = \binom{N+n}{n}^2$$

into (13) (see [10]). Changing the variable of summation gives

$$\begin{aligned} \mathscr{P}\mathscr{B}_n(t) &= \sum_{j=0}^{n} \left( \sum_{k=0}^{\infty} \binom{k+2n}{2n} t^{2k} \right) \binom{n}{j} \binom{n}{n-j} t^{2n-2j} \\ &= \frac{1}{(1-t^2)^{2n+1}} \sum_{j=0}^{n} \binom{n}{n-j} \binom{n}{j} t^{2j} \\ &= \frac{1}{(1-t^2)^{2n+1}} \sum_{j=0}^{n} \binom{n}{j}^2 t^{2j} \\ &= J(\Sigma_n \times \Sigma_n, t^2). \end{aligned}$$

Thus the $h$ vector of $\Sigma_n \times \Sigma_n$ is *independent* of the triangulation $\Delta_n$.

## 4. A $\mathbb{C}^*$ ACTION AND THE HARMONIC OSCILLATOR

In this section we explain the relation between the normal form of a perturbed $n$ dimensional harmonic oscillator and the algebra $\mathscr{B}_n$ of $\mathbb{C}^*$ invariant functions.

Consider the Hamiltonian function

$$\mathscr{H}_2(\xi, \eta) = \tfrac{1}{2} \sum_{i=0}^{n} (\eta_i^2 + \xi_i^2)$$

on the symplectic vectorspace $(T^*\mathbb{R}^{n+1}, \omega)$. $\mathscr{H}_2$ is the Hamiltonian of the $n$ dimensional harmonic oscillator. The corresponding Hamiltonian vectorfield $X_{\mathscr{H}_2}$ is

$$\dot{\xi} = \eta$$

$$\dot{\eta} = -\xi$$

whose flow is an $S^1$ action on $T^*\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1})^*$ given by

$$(14) \qquad t \cdot (\xi, \eta) = (\cos t\xi + \sin t\eta, \; -\sin t\xi + \cos t\eta).$$

We now say what is meant by the normal form of a formal power series perturbation of the harmonic oscillator. Let $P_m = P_m(T^*\mathbb{R}^{n+1}, \mathbb{R})$ be the space of homogeneous polynomials of degree $m$ on $T^*\mathbb{R}^{n+1}$. The formal power series Hamiltonian

$$\mathscr{H}(\xi, \eta) = \mathscr{H}_2(\xi, \eta) + \cdots + \mathscr{H}_m(\xi, \eta) + \cdots,$$

where $\mathscr{H}_m \in P_m$, is in normal form with respect to $\mathscr{H}_2$ iff for every $m \geqq 2$

$$L_{\mathscr{H}_2} \mathscr{H}_m = 0,$$

where $L_{\mathscr{H}_2}$ is the Lie derivative with respect to the vectorfield $X_{\mathscr{H}_2}$. In other words, $\mathscr{H}$ is in normal form iff it is invariant under the linear $S^1$ action given by (14). Because $X_{\mathscr{H}_2}$ is a semisimple linear mapping, $L_{\mathscr{H}_2}$ is a semisimple linear mapping of $P_m$ into itself. Therefore

$$(15) \qquad P_m(T^*\mathbb{R}^{n+1}, \mathbb{R}) = \ker L_{\mathscr{H}_2} | P_m \oplus \operatorname{im} L_{\mathscr{H}_2} | P_m.$$

Using (15), standard techniques (see (1)) construct a near identity formal power series symplectic mapping which brings the formal power series $\mathscr{H}$ into normal form.

To describe the normal form of formal power series perturbations of the harmonic oscillator we need to know the generators of the algebra of polynomials which are invariant under the $S^1$ action given by the flow of the harmonic oscillator. Toward this goal we introduce complex conjugate coordinates

$$z_l = \xi_l + i\eta_l \quad w_l = \xi_l - i\eta_l \quad l = 0, \ldots, n$$

on $T^*\mathbb{R}^{n+1}$. Then $\mathscr{H}_2$ becomes the quadratic Hermitian polynomial

$$\tilde{\mathscr{H}}_2 = \tfrac{1}{2} \sum_{l=0}^{n} z_l w_l$$

and $X_{\mathscr{H}_2}$ becomes $X_{\tilde{\mathscr{H}}_2}$:

$$\dot{z}_l = -2i \frac{\partial \tilde{\mathscr{H}}_2}{\partial w_l} = -iz_l \text{ for } l = 0, \ldots, n$$

$$\dot{w}_l = 2i \frac{\partial \tilde{\mathscr{H}}_2}{\partial z_l} = iw_l \text{ for } l = 0, \ldots, n.$$

The flow of $X_{\tilde{\mathscr{H}}_2}$ generates an $S^1=\{t\in\mathbb{C}^*||t|=1\}$ action on the real $2n+2$ dimensional subspace $W_{2n+2}=\{(x,y)\in\mathbb{C}^{n+1}\times(\mathbb{C}^{n+1})^*|x=z, y=\bar{z}\}$:

$$(16)\qquad t\cdot(z,w)=(tz,\bar{t}w).$$

Let $HP_m$ be the set of Hermitian polynomials of degree $m$, that is, all expressions of the form

$$\sum_{|j|+|k|=m} c_{jk}z^jw^k$$

where $c_{jk}=\bar{c}_{jk}$ and $c_{jk}\in\mathbb{C}$. Since the mapping

$$\mathscr{H}_m(\xi,\eta)\mapsto\mathscr{H}_m\left(\frac{1}{2}(z+w),\frac{1}{2i}(z-w)\right)$$

is a bijective real linear mapping between $P_m$ and $HP_m$, $\ker L_{\mathscr{H}_2}|P_m$ is in bijective correspondence with $\ker L_{\tilde{\mathscr{H}}_2}|HP_m$. Elements of $\ker L_{\tilde{\mathscr{H}}_2}$ are those Hermitian polynomials which are invariant under the $S^1$ action (16). If we restrict the $\mathbb{C}^*$ action on $T^*\mathbb{C}^{n+1}=\mathbb{C}^{n+1}\times(\mathbb{C}^{n+1})^*$ given by

$$t\cdot(x,y)=(tx,t^{-1}y)$$

to $S^1\subseteqq\mathbb{C}^*$ and to the real subspace $W_{2n+2}$ we obtain the $S^1$ action given by (16). Restricting the generators $\pi_{ij}=x_iy_j$, $0\leqq i,j\leqq n$ of the algebra $\mathscr{B}_n$ of the $\mathbb{C}^*$-invariant polynomials to $W_{2n+2}$ gives the generators $\sigma_{ij}=z_iw_j$, $0\leqq i,j\leqq n$ of the algebra of $S^1$-invariant Hermitian polynomials. Thus Re $\sigma_{ij}$ and Im $\sigma_{ij}$ are the generators of the algebra $P^{S^1}$ of polynomials on $T^*\mathbb{R}^{n+1}$ which are invariant under the flow of the harmonic oscillator.

## 5. APPENDIX. $\mathscr{A}_n$ IS ISOMORPHIC TO $\mathscr{B}_n$

First we show that the ring of polynomials $\mathscr{A}_n$ of the variety $M_{n+1,1}$ of all $(n+1)\times(n+1)$ complex matrices whose rank is less than or equal to one is isomorphic to $\mathscr{C}_n=\mathbb{C}[\mathbb{C}^{(n+1)^2}]/I_2$ where $I_2$ is the ideal generated by the $2\times 2$ minors of a generic $(n+1)\times(n+1)$ matrix. Then we show that the mapping

$$\phi:\mathbb{C}^{n+1}\times(\mathbb{C}^{n+1})^*\to M_{n+1,1}\subseteqq\mathbb{C}^{(n+1)^2}:(x,y)\mapsto x\otimes y$$

induces an isomorphism

$$\phi^*:\mathscr{C}_n\mapsto\mathscr{B}_n$$

where $\mathscr{B}_n$ is the algebra of $\mathbb{C}^*$ invariant polynomials on $\mathbb{C}^{n+1}\times(\mathbb{C}^{n+1})^*$.

Consider the algebra of polynomials $\mathbb{C}[\mathbb{C}^{(n+1)^2}]$ on the set $\mathbb{C}^{(n+1)^2}$ of $(n+1)\times(n+1)$ complex matrices. $\mathbb{C}[\mathbb{C}^{(n+1)^2}]$ is generated by the linear polynomials $\{E^*_{ij}\}$ which is the dual basis corresponding to the standard basis $\{E_{ij}\}$ of $\mathbb{C}^{(n+1)^2}$. (Here $E_{ij}$ is the matrix with 1 in the $(i,j)^{\text{th}}$ entry and 0 in the other entries). The ideal $I_2$ is generated by the polynomials

$$E^*_{ij}E^*_{kl}-E^*_{il}E^*_{kj}\quad 0\leqq i,j,k,l\leqq n.$$

The variety $M_{n+1,1}$ is defined as the set of zeroes of $I_2$. Consider the mapping

$$\psi : \mathbb{C}[\mathbb{C}^{(n+1)^2}] \to \mathscr{A}_n : f \mapsto f \,|\, M_{n+1,1}.$$

By definition of the ring of polynomial functions $\mathscr{A}_n$ on $M_{n+1,1}$, $\psi$ is a surjective ring homomorphism. To show that $\mathscr{A}_n$ is isomorphic to $\mathscr{C}_n$, we need only show that the kernel of $\psi$ is $I_2$. Suppose that $f \in \ker \psi$, then $f \,|\, M_{n+1,1} = 0$. Therefore by Hilbert's Nullstellensatz, there is a nonnegative integer $r$ such that $f^r \in I_2$. But $I_2$ is a radical ideal. (This is a well known consequence of the fact that $\mathscr{C}_n$ has a straightening law, see [2, p. 77] and [6]. For completeness we prove this in lemma 1 below). Therefore $f \in I_2$. Clearly $I_2 \subseteqq \ker \psi$. Thus $\ker \psi = I_2$, which is what we wished to show.

LEMMA 1. $I_2$ is a radical ideal.

PROOF. Denote the linear polynomial $E_{ij}^*$ on $\mathbb{C}[\mathbb{C}^{(n+1)^2}]$ by the bracket $(i, j)$. Since $\mathbb{C}[\mathbb{C}^{(n+1)^2}]$ is generated by $\{E_{ij}\}$, every element of $\mathbb{C}[\mathbb{C}^{(n+1)^2}]$ is given by a bracket polynomial. Since $I_2$ is generated by $E_{ij}^*E_{kl}^* - E_{il}^*E_{kj}^*$, every element of $\mathscr{C}_n$ is given by a bracket polynomial where the sygyzy

$$(i, j)(k, l) = (i, l)(k, j)$$

holds. Hence we may apply the straightening process of § 2 to the bracket polynomials of $\mathscr{C}_n$.

Suppose that

$$(17) \qquad T = \sum_{i=0}^{M} c_i T_i$$

is a polynomial in $\mathscr{C}_n$ with monomials $T_i$ such that $T^m = 0$. Then using the straightening process we may suppose that each $T_i$ in (17) is standard and $T_1 < T_2 < \cdots < T_M$ (where $T \leq S$ if $ht(T) < ht(S)$ or if $ht(T) = ht(S)$, then $T/ht(T) \leq S/ht(S)$. Now

$$T^m = \sum \frac{c_{i_1} \cdots c_{i_M}}{i_1! \cdots i_M!} T_1^{i_1} \cdots T_M^{i_M}.$$

Since $T_1 < \cdots < T_M$, $T_M^m$ is greater than the straightened form of $T_1^{i_1} \cdots T_M^{i_M}$ for all $1 \leqq i_1, \ldots, i_M \leqq m$ with $(i_1, \ldots, i_M) \neq (M, \ldots, M)$. Consequently

$$(18) \qquad T^m = c_M^m T_M^m + \text{lower standard monomials.}$$

Since $T^m = 0$, by linear independence of the standard monomials it follows from (18) that $c_M = 0$. Hence after finitely many repetitions of the above argument we obtain $T = 0$. Thus $I_2$ is a radical ideal, which is what we wished to prove. ■

Now consider the mapping

$$\phi : \mathbb{C}^{n+1} \times (\mathbb{C}^{n+1})^* \to \mathbb{C}^{(n+1)^2} : x, y \to x \otimes y.$$

Since $(x \otimes y)(u) = y(u)x$ for all $u \in \mathbb{C}^{n+1}$, the image of $\phi$ is a subset of $M_{n+1,1}$. Moreover $x \otimes y$ has as its matrix representative the $(n+1) \times (n+1)$ matrix $(x_i y_j)$. The following lemma gives the essential properties of $\phi$.

LEMMA 2.
(a) $\phi$ is surjective;
(b) for $A \in M_{n+1,1}$, the fiber of $\phi^{-1}(A)$ is a single $\mathbb{C}^*$ orbit, if $A \neq 0$ or is $(\mathbb{C}^{n+1} \times \{0\}) \cup (\{0\} \times (\mathbb{C}^{n+1})^*)$, if $A = 0$.

PROOF. To prove (a) suppose that $A$ has rank equal to zero. Then $A = 0$. Hence $\phi(0, y) = 0 \otimes y = 0$ does the job. Now suppose that $A$ is a nonzero element of $M_{n+1,1}$. Let $x$ be a nonzero vector in im $A$. Then $v = Ax \in \text{im } A$. If $v = 0$, then there is a nonzero vector $u$ such that

(19) $\quad x = Au.$

Hence $u \notin \ker A$. Since $\dim \ker A = n+1 - \dim \text{im } A = n+1-1$, there is a $y \in \mathbb{C}^{n+1*}$ such that $y(u) = 1$ and $\{\lambda u | \lambda \in \mathbb{C}\} \oplus \ker A = \mathbb{C}^{n+1}$. Therefore $A = x \otimes y$ because $(x \otimes y)(u) = y(u)x = x = Au$, $x \otimes y | \ker A = 0 = A | \ker A$ and $\{\lambda u | \lambda \in \mathbb{C}\} \oplus \ker A = \mathbb{C}^{n+1}$. If $v \neq 0$, then there is a nonzero $\mu \in \mathbb{C}$ such that $v = \mu x$, since $\dim \text{im } A = 1$. Therefore $x \notin \ker A$. Hence there is a $y \in (\mathbb{C}^{n+1})^*$ such that $\ker y = \ker A$ and $y(x) = \lambda$. Because

$$(x \otimes y)(x) = y(x)x = \lambda x = Ax \text{ and } (x \otimes y)|\ker A = 0 = A|\ker A,$$

$A = x \otimes y$. Therefore $\phi$ is surjective.

To prove (b) suppose that $x \otimes y = 0$. Then for all $u \in \mathbb{C}^{n+1}$, $0 = (x \otimes y)(u) = = y(u)x$. If $x = 0$, then $y \in (\mathbb{C}^{n+1})^*$ is arbitrary. If $x \neq 0$, then $y(u) = 0$ for all $u \in \mathbb{C}^{n+1}$ that is, $y = 0$. Therefore

$$\phi^{-1}(0) = (\mathbb{C}^{n+1} \times \{0\}) \cup (\{0\} \times (\mathbb{C}^{n+1})^*).$$

Now suppose that $A \in M_{n+1,1} - \{0\}$. To show that the fiber $\phi^{-1}(A)$ is a single $\mathbb{C}^*$ orbit, we analyze the freedom in the choices made in the proof of (a). If $v = 0$, then all possible choices of $u$ satisfying (19) are $u + w$ with $w \in \ker A$. But $y(u+w) = y(u) = 1$. Thus $y \in (\mathbb{C}^{n+1})^*$ does not depend on the choice of $u$. In other words, for a fixed $A \in M_{n+1,1}$, $y$ depends only on the choice of basis vector $x \in \text{im } A$. There are only $\mathbb{C}^*$ such choices. Hence $\phi^{-1}(A)$ is topologically a $\mathbb{C}^*$. But $\phi^{-1}(A)$ is a union of $\mathbb{C}^*$ orbits. Hence $\phi^{-1}(A)$ is a single $\mathbb{C}^*$ orbit. If $v \neq 0$, then again $y$ is uniquely determined by the choice of $x$. Hence $\phi^{-1}(A)$ is a $\mathbb{C}^*$ orbit. This proves (b). ■

We are now in position to show that $\mathscr{A}_n$ and $\mathscr{B}_n$ are isomorphic. The mapping $\phi$ induces a ring homomorphism

$$\phi^* : \mathscr{A}_n \to \mathbb{C}[\mathbb{C}^{n+1} \times (\mathbb{C}^{n+1})^*] : f \mapsto f \circ \phi = F.$$

Since $\phi$ is surjective, $\phi^*$ is injective. Actually, the image of $\phi^*$ is contained in $\mathscr{B}_n$. For, let $\gamma$ be the $\mathbb{C}^*$ orbit which intersects either $U = \mathbb{C}^{n+1} \times \{0\}$ or

$V=\{0\}\times(\mathbb{C}^{n+1})^*$. Since $U$ and $V$ are invariant under the $\mathbb{C}^*$ action, $\gamma\subseteqq U$ or $\gamma\subseteqq V$. Therefore $F(\gamma)\subseteqq F(U\cup V)\subseteqq F(\phi^{-1}(0))=\{f(0)\}$, that is, $F$ is constant on the $\mathbb{C}^*$ orbit $\gamma$. Now suppose that $\gamma$ is a $\mathbb{C}^*$ orbit which does not intersect $U\cup V$. Then $\gamma\subseteqq(\mathbb{C}^{n+1}-\{0\})\times((\mathbb{C}^{n+1})^*-\{0\})$. Since $\gamma=\phi^{-1}(A)$ for some $A\in M_{n+1,1}-\{0\}$, $F(\gamma)=F(\phi^{-1}(A))=\{f(A)\}$. Therefore $F$ is constant on all $\mathbb{C}^*$ orbits. In other words $F$ is a $\mathbb{C}^*$ invariant polynomial, i.e. $F\in\mathscr{B}_n$. To show that $\phi^*$ maps $\mathscr{A}_n$ onto $\mathscr{B}_n$, we recall from § 1 that the quadratic polynomials $\pi_{ij}=x_iy_j$ $0\leqq i,j\leqq n$ on $\mathbb{C}^{n+1}\times\mathbb{C}^{n+1*}$ generate $\mathscr{B}_n$. Since

$$\begin{aligned}(\phi^*)(E_{ij}^*)(x,y)&=(e_i\otimes e_j^*)^*(\phi(x,y))\\&=(e_i^*\otimes e_j)(x\otimes y)\\&=e_i^*(x)y(e_j)\\&=x_iy_j=\pi_{ij}(x,y),\end{aligned}$$

$\phi^*$ is surjective. Thus $\mathscr{A}_n$ is isomorphic to $\mathscr{B}_n$. ■

An important consequence of this isomorphism is the fact that

$$(20)\qquad \left\{\begin{aligned}0&=\phi^*(I_2)\\&=\phi^*(E_{ij}^*E_{kl}^*-E_{il}^*E_{kj}^*)\\&=\pi_{ij}\pi_{kl}-\pi_{il}\pi_{kj}\end{aligned}\right.$$

are the only relations among the generators $\{\pi_{ij}\}$ of $\mathscr{B}_n$. For suppose that $F=0$ is a polynomial relation among the generators of $\mathscr{B}_n$. Then $F=\phi^*(f)$ for some $f$ in $\mathscr{A}_n$. Hence $f=0$. This is a polynomial relation in $\mathscr{A}_n$. Therefore $f\in I_2$ since $\mathscr{A}_n=\mathscr{C}_n$. Consequently there are $a_{ijkl}\in\mathscr{A}_n$ such that

$$f=\sum a_{ijkl}(E_{ij}^*E_{kl}^*-E_{il}^*E_{kj}^*).$$

Hence

$$F=\phi^*(f)=\sum\phi^*(a_{ijkl})(\pi_{ij}\pi_{kl}-\pi_{il}\pi_{kj}).$$

Thus $F=0$ follows from (20).

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